DEGREES OF EFFICIENCY AND DEGREES OF MINIMALITY

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Abstract. In this work we characterize different types of solutions of a vector optimization problem by means of a scalarization procedure. Usually different scalarizing functions are used in order to obtain the various solutions of the vector problem. Here we consider different kinds of solutions of the same scalarized problem. Our results allow us to establish a parallelism between the solutions of the scalarized problem and the various efficient frontiers: stronger solution concepts of the scalar problem correspond to more restrictive notions of efficiency. Besides the usual notions of weakly efficient and efficient points, which are characterized as global (or strict global) solution of the scalarized problem, we also consider some restricted notions of efficiency, such as strict and proper efficiency, which are characterized as Tikhonov well-posed minima and sharp minima for the scalarized problem.

Key words. Vector Optimization, scalarization, proper efficiency, strict efficiency, sharp minima, well-posed minima

AMS subject classifications. 90C29, 90C31, 90C48, 49K40

1. Introduction. Both theory and practice of Vector Optimization have always been closely related to scalarization procedures. The most widely used is probably the linear scalarization; for Pareto optimization problems, in which the outcome space is ordered componentwise, it consists in the consideration of a weighted sum of components of the objective function with nonnegative coefficients.

In the seminal paper by Kuhn and Tucker [21] for the first time the concept of proper efficiency was introduced, precisely in order to prove that the multipliers of all components of the objective function in the necessary optimality conditions be (strictly) positive. To reach this result one has to require, besides a constraint qualification, a further requirement on the efficient point to avoid anomalous features.

In this fashion properly efficient points can be characterized (under convexity assumptions) as the solutions of a linearly scalarized problem in which all coefficients are positive. From a geometric point of view this entails that an open halfplane exists, whose normal vector is strictly positive and is disjoint from the image set.

In the last fifty years the notion of proper efficiency has been described in a number of ways (see e.g. [13, 27, 6] for references and comparisons); the various definitions emphasize different aspects (boundedness of trade-offs between objectives, disjointness between the ordering cone and some conical approximation of the image set, stability properties with respect to the ordering structure), but they can be seen as an extension of the primitive idea in that they can be geometrically described in terms of separation between the ordering cone and the image set by means of an open convex cone or an open convex set.

It is thus evident that the nature of proper efficiency, whose origin has a purely local nature (in the classical paper by Kuhn and Tucker its definition is given in terms of the derivatives of the objective and constraint functions), also entails a global character.

The deep connection with scalarization has always been stressed and various ad hoc scalarization techniques have been devised to relate, in a nonconvex problem, proper efficiency according to the various definitions to an optimal scalar solution.

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For instance in the scalarization approach proposed by Jahn [17, 18], which consists in the minimization of the distance (according to an appropriate parametric norm) from some ideal point, two different families of norms are used to characterize efficient and properly efficient point, the former being related to the ordering cone and the latter to another cone which contains the ordering cone in its interior; more generally, to obtain more restrictive notions of efficiency as minimal solution of a scalarized problem, stronger monotonicity features must be imposed on the scalarizing function (see for instance [27, 6, 10, 28]).

Proceeding in this vein one finds that efficient and properly efficient points can be seen as minimal solutions of different scalar problems.

A different type of restriction on efficiency has been posed in order to control the asymptotic behaviour of unbounded admissible sequences, thus obtaining the notion of strict efficiency [2].

The aim of this work is to characterize the various sets of solutions of a vector optimization problem by means of a unique scalarizing function. A solution belonging to a more restrictive set of solution is found as an optimal scalar solution according to a more restrictive definition of minimality: we will consider strict (i.e. unique) minima, sharp minima or others. In particular we will refer to some growth conditions known in scalar optimization and to the notion of Tykhonov well-posedness.

In Section 2 we will introduce six different types of solutions of a vector optimization problem and show their relationships. Section 3 is devoted to the scalarizing function; we point out that our scalarizing function (the \( \Delta \) function, introduced by Hiriart-Urruty [14, 15] to obtain nonsmooth optimality conditions) is a very simple tools which immediately emphasizes the geometry of the ordering relation and has a very simple form in specific problems, thus allowing for simple calculations. In Section 4 the main results of the paper are given: the six types of efficient solutions presented above appear as solutions of the scalarized problem with increasingly more restrictive minimality properties. In Section 5 we consider again the concepts introduced in Section 2 with the aim to compare them to other, maybe better known, notions of efficiency. Some equivalence results are proved there in infinite and finite dimensional spaces, which complement the results in [13].

2. Degrees of Efficiency. Let \( Y \) be a normed space, \( S \subset Y \) be the set of admissible points and \( K \subset Y \) a closed, convex, pointed cone inducing on \( Y \) the partial order relation given by \( y_1 \geq y_2 \) if and only if \( y_1 - y_2 \in K \). We say that the set \( \Theta \subset Y \) is a base for \( K \) if \( \Theta \) is convex with \( 0 \not\in \text{cl} \Theta \) and \( K \setminus \{0\} = \text{cone} \Theta = \{y \in Y : y = \lambda \theta, \lambda > 0, \theta \in \Theta\} \), i.e. \( K \) is the cone generated by \( \Theta \). Here and in the sequel, we denote with \( \text{cl} A \), \( \text{int} A \), \( \partial A \) and \( A^c \) the closure, the interior, the boundary and, respectively, the complement of the set \( A \subseteq Y \). Moreover we will denote with \( B \) (respectively \( \hat{B} \)) the closed (resp. open) unit ball in \( Y \) and with \( d_A(y) = \inf\{\|a - y\|, a \in A\} \) the distance function to the set \( A \subseteq Y \).

**Definition 2.1.** A point \( y_0 \in S \) is said to be **efficient** (w.r.t. \( K \)) and denoted \( y_0 \in E(S, K) \) (or \( y_0 \in E(S) \) if no confusion arises) if
\[
(S - y_0) \cap -K = \{0\}.
\]

**Definition 2.2.** In the case where \( \text{int} K \neq \emptyset \), a point \( y_0 \in S \) is said to be **weakly efficient** (denoted \( y_0 \in WE(S) \)) if
\[
(S - y_0) \cap -\text{int} K = \emptyset.
\]
Various other conditions have been used to single out particular classes of efficient points with special features. The concept of strict efficiency was introduced in [2] in order to obtain upper semicontinuity of the section mapping $G(y) = S \cap (y - K)$ at an efficient point.

**Definition 2.3.** A point $y_0 \in S$ is called strictly efficient (denoted $y_0 \in \text{StE}(S)$), if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$(S - y_0) \cap (\delta B - K) \subseteq \varepsilon B.$$  

An efficient point $y_0 \in S$ is strictly efficient if the points $y \in Y$ which are dominated from $y_0$, i.e. $y \leq y_0$, and are bounded away from $y_0$, cannot be arbitrarily close to the feasible region $S$.

Proper efficiency instead implies a control on the tangentially admissible directions close to the efficient point. It has been defined in a great number of ways; in [13] an attempt was made to classify the various definitions along three main classes whose elements are proved to be equivalent in Euclidean spaces. Further notions are analysed in the paper by Zalinescu [27]. It readily becomes evident that most definitions of proper efficiency entail a restriction on the admissible region which has a global character. In the sequel we will introduce a few of them; since the ones we will mention are often not the better known, we give in Section 5 some comparisons with others, without any scope of completeness. The interested reader can refer to [6, 27, 13] for more details.

**Definition 2.4.** [6] The point $y_0 \in S$ is called superefficient ($y_0 \in \text{SE}(S)$) if there exists $M > 0$ such that

$$(2.1) \quad \text{cl cone}(S - y_0) \cap (B - K) \subseteq MB.$$  

**Remark 2.5.** The inclusion (2.1) can be equivalently expressed (see [6]) as

$$\text{cone}(S - y_0) \cap (B - K) \subseteq MB;$$  

in the sequel we will often use this equivalent formulation.

**Definition 2.6.** The point $y_0 \in S$ is locally superefficient ($y_0 \in \text{LSE}(S)$) if it is efficient and there exist $\eta > 0$ and $M > 0$ such that

$$\text{cl cone}((S - y_0) \cap \eta B) \cap (B - K) \subseteq MB.$$  

It is obvious that $y_0 \in \text{LSE}(S)$ if and only if there exists $\eta > 0$ such that $y_0 \in \text{SE}(S \cap (y_0 + \eta B))$.

We will also refer to another notion of efficiency which is somehow intermediate between the previous ones.

**Definition 2.7.** The point $y_0 \in S$ is called tightly properly efficient ($y_0 \in \text{TP E}(S)$) if there exists an open convex set $C \subset Y$ with $0 \in \partial C$, satisfying $(S - y_0) \cap C = \emptyset$ and there exists $\delta > 0$ such that

$$(2.2) \quad C^\circ \cap (\delta B - K) \subseteq B.$$
Our first aim is to clarify the relationships among the above concepts.

To understand the relation between superefficiency and tight efficiency, we will need the concept of dilating cone. If \( \Theta \) is a base for the cone \( K \), then by definition \( 0 \notin \text{cl} \Theta \) and \( \| \theta \| > \delta > 0 \) for some \( \delta > 0 \) and all \( \theta \in \Theta \). For every \( \varepsilon < \delta \) the cone \( K_\varepsilon = \text{cone}(\Theta + \varepsilon B) \), where \( B \) is the open unit ball in \( Y \), is an open convex cone containing \( K \setminus \{0\} \). Note that the assumption that a cone \( K \) has a base is equivalent to the existence of strictly positive continuous linear functionals, i.e. the set

\[
K^{+i} = \{ \ell \in Y^\prime : \ell(k) > 0, \forall k \in K \setminus \{0\} \}
\]

is nonempty, where \( Y^\prime \) is the (topological) dual space of \( Y \). Moreover if a closed cone \( K \) admits a base then the base can be taken to be closed.

**Theorem 2.8.** If the cone \( K \) has a bounded base, then \( SE(S) \subseteq TPE(S) \), i.e. every superefficient point is also tightly efficient.

**Proof.** To simplify the presentation and without loss of generality, we will prove the result under the assumption that \( y_0 = 0 \). As the first step of the proof, we need to show that, if \( 0 \) is superefficient, there exists an open convex dilating cone \( K_\varepsilon \) such that

\[
\text{cl cone } S \cap -K_\varepsilon = \emptyset.
\]

This is exactly the content of Proposition 3.3 in [6].

Secondly we prove that there exists \( N > 0 \) such that

\[
(B - K) \cap (-K_\varepsilon)^c \subseteq NB.
\]

Indeed set \( W = (-K_\varepsilon)^c \) and suppose relation (2.3) is false, i.e. there exists a sequence \( w_n, n \in \mathbb{N}, \) such that \( w_n \in W, \|w_n\| > n \) and \( w_n \in B - K \). The latter means that there exists a sequence \( k_n \in K \) such that \( \|w_n + k_n\| \leq 1 \). Thus there exists a sequence \( b_n \in B \) such that \( w_n + k_n = b_n \), which can be rewritten as

\[
w_n = b_n - k_n = b_n - \lambda_n \theta_n = \lambda_n(b_n/\lambda_n - \theta_n),
\]

where \( \theta_n \in \Theta \) and \( \lambda_n > 0 \). But this is a contradiction to \( w_n \notin (-K_\varepsilon)^c \), since, by the boundedness of \( \Theta \), we have that \( \lambda_n \to +\infty \) as \( \|w_n\| \to +\infty \) and \( 1/\lambda_n \) becomes smaller than any fixed \( \varepsilon \). Thus eventually \( w_n \in \text{cone}(\varepsilon B - \Theta) \).

Now the definition of tight efficiency is verified with \( C = -K_\varepsilon \), by taking \( \delta = 1/N \).

\( \Box \)

Cones with a bounded base have been extensively used in the literature related to ordered spaces and Vector Optimization under a variety of different characterizations which were later proved to be equivalent. For instance in [26] is proved that a cone \( K \) has a bounded base if and only if it is supernormal (or nuclear) (see [16] for the definition) and if and only if there exists \( \phi \in K^{+i} \) and a scalar \( \alpha \) such that \( \phi(k) \geq \alpha \|k\| \) for all \( k \in K \) (Bishop-Phelps cone) and in [3] is proved that a cone \( K \) has bounded base if and only if it allows plastering (see [20]).

To show that a tightly efficient point is both locally superefficient and strictly efficient, we need a preliminary result.

**Lemma 2.9.** The point \( y_0 \) is tightly efficient in \( S \) if and only if there exists an open convex set \( C \subset Y \), with \( 0 \in \partial C \) satisfying \( (S - y_0) \cap C = \{0\} \) and for every \( \varepsilon > 0 \) there exists \( \delta' > 0 \) such that

\[
C^c \cap (\delta'B - K) \subseteq \varepsilon B.
\]
Proof. Possibly after a translation of $S$, assume that $y_0 = 0$ is a tightly properly efficient point for $S$ and fix $\varepsilon > 0$. If $\varepsilon \geq 1$ the thesis follows with $\delta' = \delta$, as in Definition 2.7. So let $\varepsilon \in (0, 1)$ and choose $\delta$ such that

$$(\delta B - K) \cap C^c \subseteq B.$$  

This implies that

$$\varepsilon(\delta B - K) \cap \varepsilon(C^c) \subseteq \varepsilon B.$$  

Noting that $\varepsilon(\delta B - K) = (\varepsilon \delta B - \varepsilon K)$, the ‘only if’ part of the thesis follows by the observation that for an open convex set $C$ with $0 \in \text{cl} C$ and any $\varepsilon \in (0, 1)$ it holds $\varepsilon C \subseteq C$ and $C^c \subseteq (\varepsilon C)^c = \varepsilon(C^c)$. The reverse inclusion is obvious. \Box

**Theorem 2.10.** If the point $y_0$ is tightly efficient in $S$, then it is both strictly efficient and locally superefficient.

**Proof.** Strict efficiency follows by simply comparing the definitions and using Lemma 2.9. Moreover, if $(\delta B - K) \cap C^c \subseteq \varepsilon B$, then it holds $-K \setminus \{0\} \subseteq C$. Indeed if $k \in K \setminus \{0\}$ and $-k \notin C$, then $-\alpha k \notin C$ for all $\alpha > 1$ (remember that $0 \in \text{cl} C$) and $-\alpha k \in -K \subseteq \delta B - K$ and thus we have a contradiction.

To prove that $0 \in LSE(S)$ we first notice that

$$S \cap C = \emptyset \Rightarrow S \subset C^c \Rightarrow [1, \infty) \cdot S \subset C^c.$$  

Hence cone $(S \cap B) \subset B \cup [1, \infty) \cdot S \subset B \cup C^c$. Because $(\delta B - K) \cap C^c \subset B$ (for some $\delta > 0$), we have

$$(B - K) \cap C^c \subset (\delta B - K) \cap C^c \subset B$$  

when $\delta \geq 1$ and

$$(B - K) \cap C^c \subset (B - K) \cap \delta^{-1} C^c \subset \delta^{-1} B$$  

when $\delta < 1$. Thus, taking $M = \max\{1, \delta^{-1}\}$, we get

$$\text{cone}(S \cap B) \subset (B \cap (B - K)) \cup (C^c \cap (B - K)) \subset MB. \Box$$

The opposite relation is true under the assumption that the cone $K$ has a bounded base.

**Theorem 2.11.** If the cone $K$ has a bounded base and the point $y_0$ is both strictly efficient and locally superefficient in $S$, then it is also tightly efficient.

**Proof.** Take $\eta$ such that the point $y_0$ is superefficient for the admissible set $S \cap (y_0 + \eta B)$ and, by strict efficiency, choose $\delta'$ such that

$$(S - y_0) \cap (\delta' B - K) \subseteq \eta B.$$  

Then follow the proof of Theorem 2.8 to show that there exists $\varepsilon > 0$ such that

$$(S - y_0) \cap \eta B \cap (-K_{\varepsilon}) = \emptyset$$  

(see again [6, Prop. 3.3]) and that there exists $N > 0$ such that

$$(B - K) \cap (-K_{\varepsilon})^c \subseteq NB$$
which is equivalent to
\[
[1/N)B - K] \cap (-K_x)^c \subseteq B.
\]
Now choose \( \delta = \min(\delta', 1/N) \) and set \( C = (\delta \hat{B} - K) \cap (-K_x) \), where \( \hat{B} = \text{int} B \) denotes the open unit ball. Then \( C \) is open and convex, with \( 0 \in \partial C \). It follows from (2.4) that
\[
(\delta B - K) \cap (S - y_0) \subseteq \eta B,
\]
which, together with (2.5), yields
\[
(\delta B - K) \cap (S - y_0) \cap -K = \emptyset,
\]
that is
\[
(S - y_0) \cap C = \emptyset.
\]
Moreover
\[
\left[ \frac{\delta}{2} B - K \right] \cap C^c = \left[ \frac{\delta}{2} B - K \right] \cap [((\delta \hat{B} - K)^c \cup (-K_x)^c] = \left[ \frac{\delta}{2} B - K \right] \cap (-K_x)^c \subseteq B,
\]
which is our thesis.

To summarize we can state that, if the cone \( K \) has a bounded base, then the following inclusions hold:
\[
SE(S) \subseteq TPE(S) = LSE(S) \cap StE(S) \subseteq LSE(S) \cup StE(S) \subseteq E(S) \subseteq W E(S)
\]
and simple examples in \( IR^2 \) show that all inclusions are strict. The six notions we introduced can thus be seen as six different degrees of efficiency for a vector optimization problem.

3. Scalarization. The theory and the methods of scalarization have always been of the utmost importance in order to solve a vector optimization problem. The linear scalarization is historically the first method proposed and the most widely known and used; it consists in the minimization, over the set \( S \), of the function \( \lambda \cdot y \), with \( \lambda \in K^+ \). Besides this, in order to treat nonconvex problems, the method of compromise solutions and its generalizations is of great relevance. It consists in the minimization of the distance from some reference objective, often not belonging to the admissible region, but dominating all available alternatives. This was originally done, in the Pareto setting in which the ordering cone is simply the nonnegative orthant, with the use of the supremum norm of \( Y = IR^p \). The main idea has successively been extended to the setting of general ordered vector space (see e.g. [17, 18, 19] and references therein) by defining the norm as the Minkowski functional of the order interval \([-a, a] \equiv (-a + K) \cap (a - K)\), i.e.
\[
\|y\|_a = \inf \{ \lambda > 0 : \lambda^{-1} y \in [-a, a] \},
\]
where \( a \) is some point in the interior of the ordering cone \( K \) and the order interval \([-a, a]\) is a closed, convex set with nonempty interior.

For a fixed reference point \( \ell \in Y \) such that \( S \subset \ell + K \), Jahn characterizes weakly efficient (respectively efficient) points as the nearest (resp. unique nearest) points to
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ℓ with respect to the distance induced by ∥ · ∥a. To characterize properly efficient points, Jahn considers the nearest points to ℓ according to a different norm, defined by means of the dilating cone Kε.

Closely related to this approach are some variant in which the scalarizing function is defined by means of a sort of Minkowski functional of sets related to the ordering cone, as for instance in [27, 10, 28]. In this case the scalarizing functional φ : X → ℜ is given by

\[ \phi(x) = \inf\{\lambda \in ℜ : x \in \lambda e - A\}, \]

where e ∈ int A. If the set A coincides with the ordering cone K, then characterizations of efficient and weakly efficient points are obtained. But, in order to obtain properly efficient points as minimal scalar solution, the scalarizing function must be defined with respect to some convex set or cone A which contains K \{0\} in its interior and hence every properly efficient point is solution of a different scalarized problem.

The main results of this paper give a complete characterization of the different types of efficient points for a vector optimization problem by means of different degrees of minimality of a unique scalarized problem. This is obtained by means of a special scalarizing function.

**Definition 3.1.** For a set A ⊆ Y let the function \( \Delta_A : Y \to ℜ \cup \{±\infty\} \) be defined as

\[ \Delta_A(y) = d_A(y) - d_{Y \setminus A}(y), \]

with \( d_\phi(y) = +\infty \).

The function \( \Delta \) was introduced in [14, 15] to analyse the geometry of nonsmooth optimization problems and obtain necessary optimality conditions. It has later been used in [8, 1, 22, 23]. Its main properties are gathered together in the following proposition.

**Proposition 3.2.** If the set A is nonempty and \( A \neq Y \), then
1. \( \Delta_A \) is real valued;
2. \( \Delta_A \) is 1-Lipschitzian;
3. \( \Delta_A(y) < 0 \) for every \( y \in \text{int} A \), \( \Delta_A(y) = 0 \) for every \( y \in \partial A \) and \( \Delta_A(y) > 0 \) for every \( y \in \text{int} A^c \);
4. if \( A \) is closed, then it holds \( A = \{ y : \Delta_A(y) \leq 0 \} \);
5. if \( A \) is convex, then \( \Delta_A \) is convex;
6. if \( A \) is a cone, then \( \Delta_A \) is positively homogeneous;
7. if \( A \) is a closed convex cone, then \( \Delta_A \) is nonincreasing with respect to the ordering relation induced on Y, i.e. the following is true: if \( y_1, y_2 \in Y \) then

\[ y_1 - y_2 \in A \implies \Delta_A(y_1) \leq \Delta_A(y_2); \]

if \( A \) has nonempty interior, then

\[ y_1 - y_2 \in \text{int} A \implies \Delta_A(y_1) < \Delta_A(y_2). \]

**Proof.** Statements 1 – 4 and 6 are immediate. To prove 5 one can pass through a characterization (given in [15]) of \( \Delta_A \) as an infimal convolution: \( \Delta_A(y) = (\mu_A \nabla \| \cdot \|)(y) = \inf\{\mu_A(x) + \|y - x\| : x \in Y\} \), where the function \( \mu_A(y) = +\infty \) for \( y \not\in A \) and \( \mu_A(y) = -d_{Y \setminus A}(y) \) for \( y \in A \) can be proved to be convex when \( A \) is convex. This result is stated without proof in [14] and is equivalent to the concavity of the function
\(d_{A^c} \) on \(A\). To show that the latter holds when \(A\) is convex, one can notice that for any \(x, y \in A\), the closed balls \(B_x\) and \(B_y\) centered in \(x\) and in \(y\) with radii \(d_{A^c}(x)\) and \(d_{A^c}(y)\) respectively are contained in \(\text{cl} A\) (which is itself convex), as well as the set \(H = \text{conv} \{B_x \cup B_y\}\). Moreover, for every \(\alpha \in [0, 1]\), the ball centered in \(\alpha x + (1 - \alpha)y\) with radius \(\alpha d_{A^c}(x) + (1 - \alpha)d_{A^c}(y)\) is contained in \(\text{cl} A\) (which is itself convex), as well as the set \(H = \text{conv} \{B_x \cup B_y\}\). Moreover, for every \(\alpha \in [0, 1]\), the ball centered in \(\alpha x + (1 - \alpha)y\) with radius \(\alpha d_{A^c}(x) + (1 - \alpha)d_{A^c}(y)\) is contained in \(H\) and hence \(\alpha d_{A^c}(\alpha x + (1 - \alpha)y) \geq d_{A^c}(\alpha x + (1 - \alpha)y)\). The result then follows, since the convolution of a convex function with the norm is itself convex. To prove 7 use nonpositivity of \(\Delta_A\) on \(A\) and subadditivity to write \(0 \geq \Delta_A(y_1 - y_2) \geq \Delta_A(y_1) - \Delta_A(y_2)\); the second implication is proved analogously.

For our purposes, let \(A = -K\). Then the function \(\Delta_{-K}(y)\) is sublinear and nondecreasing with respect to the ordering induced by \(K\).

It is important to note that the use of the scalarizing function \(\Delta_{-K}\) implies no assumption (explicit or implicit) of boundedness of the admissible region \(S\). On the other hand such assumptions are required for most other scalarizations. For instance \(S\) has to be contained in a halfspace if the linear scalarization is used or it has to be lower bounded, in the sense that there exists an element \(\ell \in Y\) such that \(s \geq \ell\) for every \(s \in S\), if the distance from an ideal point is used.

This remark has some relevance in that the main differences among the different types of efficient points considered in our analysis disappear under boundedness assumptions.

We give now some examples of how the function \(\Delta_{-K}\) looks like for different choices of the space \(Y\) and its norm and the ordering cone \(K\).

**Examples**

1. Let \(Y = \mathbb{R}^n\) with the euclidean norm \(\| \cdot \|_2\) and \(K = \mathbb{R}^n_+\). Then it holds
   \(d_{-K}(y) = \|y^+\|\), where \(y^+_i = \max(y_i, 0)\), \(i = 1, ..., n\) and
   \[d_{Y_{-K}}(y) = \begin{cases} 0 & \text{if } y_i \geq 0 \text{ for some } i \\ -\max_i y_i & \text{if } y_i < 0 \text{ for all } i \end{cases}\]

   Thus it holds
   \[\Delta_{-K}(y) = \begin{cases} \|y^+\| & \text{if } y \notin -K \\ \max_i y_i & \text{if } y \in -K \end{cases}\]

2. The function \(\Delta_{-K}\) takes a more familiar form if we consider \(Y = \mathbb{R}^n\) with the norm \(\|y\|_\infty = \max |y_i|\); in this case we have:
   \[d_{-K}(y) = \begin{cases} \max_i y_i & \text{if } y \notin -K \\ 0 & \text{if } y \in -K \end{cases}\]

   and
   \[d_{Y_{-K}}(y) = \begin{cases} 0 & \text{if } y \notin -K \\ -\max_i y_i & \text{if } y \in -K \end{cases}\]

   and thus, for all \(y \in Y\),
   \[\Delta_{-K}(y) = \max_i y_i.\]
3. The same reasoning can be applied to the case where $Y = \mathcal{C}(T)$, the space of continuous functions defined over the compact set $T$, with the supremum norm and the ordering cone of nonnegative functions $K = \{ y \in Y : y(t) \geq 0, \forall t \in T \}$. In this case, for those $y \in \mathcal{C}(T)$ for which there exists some $t \in T$ with $y(t) \geq 0$, it holds $\Delta_{-K}(y) = d_{-K}(y) = \max_{t \in T} y(t)$ and for those $y \in \mathcal{C}(T)$ such that $y(t) < 0$ for all $t \in T$, then it holds $\Delta_{-K}(y) = -d_{Y \setminus -K}(y) = \max\{ y(t), t \in T \}$. Therefore

$$\Delta_{-K}(y) = \max_{t \in T} y(t)$$

for all $y \in Y$.

4. Analogously for the space $Y = L^\infty(I)$ of essentially bounded functions on the interval $I \subseteq \mathbb{R}$, with the usual norm and the cone $K$ of nonnegative functions, it holds

$$\Delta_{-K}(y) = \sup_{\omega \in I} y(\omega),$$

where sup means essential supremum.

5. For the spaces $L^p$, with $1 \leq p < \infty$ (with the usual norm $\| \cdot \|_p$), we have that the nonnegative orthant $K$ has empty interior. In this case we have $\Delta_{-K}(y) = d_{-K}(y) = \| y^+ \|$, where $y^+ = \sup(y, 0)$ has the usual meaning of lattice theory.

4. Characterizations. We give in this section the main results of the paper. The various types of efficient solutions introduced in Section 2 will be characterized by different degrees of minimality of the scalar solutions of the parametrized problem:

$$(P_p) \quad \min_{y \in S} \Delta_{-K}(y - p),$$

with $p \in Y$.

Theorem 4.1 characterizes efficient solutions as the strict minimal points for the scalarized problem. This result is well-known for other type of scalarizations and was proved in [22] with the scalarizing function $\Delta_{-K}$. It also appears in [23] in an axiomatic setting. We prove it for completeness.

**Theorem 4.1.** Let $y_0 \in S$ be an admissible point. Then $y_0 \in E(S)$ if and only if there exists $\tilde{y} \in Y$ such that $y_0$ is a unique (strict) global solution of $(P_{\tilde{y}})$.

**Proof.** If $y_0$ is efficient, then it is a strict minimum for $(P_{y_0})$. Indeed if $y_0 \in E(S)$ then $y - y_0 \not\in -K \setminus \{ 0 \}$ for every $y \in S$ and we have that $\Delta_{-K}(y - y_0) = d_{-K}(y - y_0)$ is positive whenever $y \neq y_0$ and null at $y = y_0$. To prove the converse, suppose that $\Delta_{-K}(y_0 - \tilde{y}) < \Delta_{-K}(y - \tilde{y})$ for all $y \in S$ with $y \neq y_0$. If there exists in $S$ some point $y_1 \neq y_0$ such that $y_1 \leq y_0$, then it would hold $y_1 - \tilde{y} \leq y_0 - \tilde{y}$ and, by Proposition 3.2-7 we get $\Delta_{-K}(y_1 - \tilde{y}) \leq \Delta_{-K}(y_0 - \tilde{y})$, which is absurd. \(\square\)

**Theorem 4.2.** Let $y_0 \in S$ be an admissible point and suppose that $\text{int} K \neq \emptyset$; then $y_0$ is weakly efficient in $S$ if and only if there exists $\tilde{y} \in Y$ such that $y_0$ is a global solution of $(P_{\tilde{y}})$.

**Proof.** The proof is analogous to the one of Theorem 4.1 and hence is omitted. \(\square\)

We observe that, if the cone $K$ has empty interior, then $\Delta_{-K} = d_{-K}$ and, for every $\tilde{y} \in S$, $d_{-K}(y - \tilde{y}) = d_{-K}(\tilde{y} - \tilde{y}) = 0$ for all $y \leq \tilde{y}$, with $y \in S$. Hence if the point $\tilde{y}$ is not itself efficient, then the scalarized problem will have solutions at all $y \in S$.\(\square\)
with \( y \leq \hat{y} \) and uniqueness will fail. In this case, Theorem 4.1 can be reformulated as follows.

**Theorem 4.3.** Let \( y_0 \in S \) be an admissible point; then \( y_0 \in E(S) \) if and only if \( y_0 \) is a unique (strict) global solution of \((P_{y_0})\).

**Proof.** Trivial \( \Box \)

In the sequel we will always use problem \((P_{y_0})\) to characterize efficiency properties of the point \( y_0 \). It is understood that, if \( \text{int } K \neq \emptyset \), then a greater generality can be achieved (as in Theorem 4.1) considering a different value \( \hat{y} \) for the parameter.

If the point \( y_0 \) is strictly efficient, then we can prove a further property for the scalarizing function \( \Delta_{-K}(\cdot - y_0) \).

**Theorem 4.4.** Let \( y_0 \in S \) be an admissible point. Then \( y_0 \) is strictly efficient in \( S \) if and only if there exists a nondecreasing function \( \gamma : \mathbb{R}^+ \to \mathbb{R}^+ \) with \( \gamma(0) = 0 \) and \( \gamma(t) > 0 \) for all \( t > 0 \), such that \( \Delta_{-K}(y - y_0) \geq \gamma(||y - y_0||) \) for all \( y \in S \).

**Proof.** Strict efficiency of \( y_0 \) can be rephrased as: for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that \( d_{-K}(y - y_0) > \delta \) for every \( y \in S \) with \( ||y - y_0|| > \varepsilon \). So suppose that the point \( y_0 \) is strictly efficient in \( S \) and consider the functions

\[
\gamma_0(\varepsilon) = \inf\{d_{-K}(y - y_0) \mid y \in S, ||y - y_0|| \geq \varepsilon\}
\]

and

\[
\gamma(\varepsilon) = \min(\gamma_0(\varepsilon), 1).
\]

It is easily seen that \( \gamma \) is nondecreasing, null at the origin and positive elsewhere; moreover, for \( y \in S \), it holds

\[
\Delta_{-K}(y - y_0) = d_{-K}(y - y_0) \geq \gamma(||y - y_0||).
\]

If, on the other hand, there exists a nondecreasing function \( \gamma \) with the above properties and such that \( \Delta_{-K}(y - y_0) \geq \gamma(||y - y_0||) \) for all \( y \in S \), then it holds \( \Delta_{-K}(y - y_0) > 0 \) for all \( y \in S \) with \( y \neq y_0 \), yielding \( y_0 \in E(S) \) and \( \Delta_{-K}(y - y_0) = d_{-K}(y - y_0) \) for all \( y \in S \). To show that \( y_0 \in \text{StE}(S) \), to every \( \varepsilon > 0 \) we can associate \( \delta = \inf\{d_{-K}(y - y_0) \mid ||y - y_0|| > \varepsilon\} \) and the result is proved. \( \Box \)

The definition of strict efficiency can be reformulated to offer a version of Tikhonov well-posedness (in the image) for a vector problem. The problem of minimizing an extended real valued function \( f \) defined over some metric space \( X \) is said to be Tikhonov well-posed when it admits a unique minimum \( x_0 \) and for every sequence \( x_n \) such that \( f(x_n) \) converges to the infimal value \( f(x_0) \), it holds \( x_n \to x_0 \). We refer to the monograph by Dontchev and Zolezzi [11] for more details about Well-Posedness in optimization. The term forcing function is used there for a function \( \gamma \) with the properties mentioned in Theorem 4.4.

Though it is well known by specialists in well-posedness and rather trivial, it should be recalled that the forcing property of a nondecreasing function \( \gamma : \mathbb{R}^+ \to \mathbb{R}^+ \) with \( \gamma(0) = 0 \) can be equivalently expressed as \( \gamma(t) > 0 \) for all \( t > 0 \) or \( \gamma(t) \to 0 \Rightarrow t_n \to 0 \).

In our formulation the objective function is \( \Delta_{-K}(\cdot - y_0) \), defined over \( Y \) and extended to \( +\infty \) outside the set \( S \). The minimization problem \((P_{y_0})\) is Tikhonov well-posed if \( \Delta_{-K}(y - y_0) > 0 \) for all \( y \in S \) with \( y \neq y_0 \) and

\[
(y_n \in S, \quad d_{-K}(y_n - y_0) \to 0) \quad \implies \quad y_n \to y_0.
\]
that is if, when a sequence \( y_n \in S \) is minimizing, i.e. it belongs to \( S \) and approaches the region dominated by \( y_0 \), then it must converge to \( y_0 \).

Theorem 4.5 makes clearer the interpretation of strict efficiency as a sort of well-posedness for the scalarized problem \((P_{y_0})\). Notice that the equivalence between Tikhonov well-posedness and the existence of a forcing function is a known results; see e.g. [29].

**Theorem 4.5.** The admissible point \( y_0 \in S \) is strictly efficient for \( S \) if and only if \( y_0 \) is a solution of \((P_{y_0})\) and the problem \((P_{y_0})\) is Tikhonov well-posed.

**Proof.** If \( y_0 \) is strictly efficient then it is the unique solution for the scalarized problem \( P_{y_0} \), and there exists a forcing function \( \gamma \) such that \( \Delta_{-K}(y-y_0) \geq \gamma(||y-y_0||) \). Hence all minimizing sequences, i.e. sequences \( y_n \in S \) such that \( \Delta_{-K}(y_n-y_0) \to 0 \), must converge to \( y_0 \), since \( \gamma(t_n) \to 0 \) implies \( t_n \to 0 \). Conversely if the problem \((P_{y_0})\) is well-posed, then \( y_0 \) is efficient and, on the set \( S \), it holds \( d_{-K} = \Delta_{-K} \). Thus we can consider the function \( \gamma(\varepsilon) = \inf \{d_{-K}(y-y_0) \mid ||y-y_0|| \geq \varepsilon \} \), it holds by construction \( d_{-K}(y-y_0) \geq \gamma(||y-y_0||) \) and it is easy to see that \( \gamma \) is nondecreasing on \([0, +\infty)\) with \( \gamma(\varepsilon) > 0 \) for \( \varepsilon > 0 \) and hence \( y_0 \) is strictly efficient. \( \square \)

As we have seen in Theorem 4.4, strict efficiency of the point \( y_0 \) guarantees the existence of a forcing function for \( \Delta_{-K}(\cdot - y_0) \). But there is no control on the slope of such function close to zero. This is where proper efficiency comes into the picture. Indeed we will show that some properties of the forcing function \( \gamma \) can be strengthened when \( y_0 \) is properly efficient to obtain a linear growth for \( \Delta_{-K} \). This will hold near \( y_0 \) if \( y_0 \) is locally superefficient and will hold globally for \( y_0 \in SE(S) \).

**Theorem 4.6.** Let \( y_0 \in S \) be an admissible point. Then it holds:

a) \( y_0 \) is superefficient in \( S \) if and only if there exists \( L > 0 \) such that \( \Delta_{-K}(y-y_0) \geq L||y-y_0|| \) for all \( y \in S \).

b) \( y_0 \) is locally superefficient in \( S \) if and only if there exist \( \eta > 0 \) and \( L > 0 \) such that \( \Delta_{-K}(y-y_0) \geq L||y-y_0|| \) for all \( y \in S \cap (y_0 + \eta B) \).

**Proof.** (a) We first need to show that, for an efficient point \( y_0 \in S \), the inclusion

\[
(4.1) \quad \text{cone}(S - y_0) \cap (B - K) \subseteq MB
\]

can be expressed by saying that for each \( y \in S \) and each \( k \in K \) it holds

\[
(4.2) \quad ||y - y_0|| \leq M||y + k - y_0||.
\]

Indeed if (4.1) does not hold, then there exist a positive number \( \lambda \) and some \( \bar{y} \in S \), \( \bar{k} \in K \) and \( \bar{b} \in B \) such that

\[
(4.3) \quad \lambda(\bar{y} - y_0) = \bar{b} - \bar{k}
\]

and

\[
\lambda||\bar{y} - y_0|| > M.
\]

From (4.3) we derive

\[
||\lambda(\bar{y} - y_0) + \bar{k}|| \leq 1
\]

and

\[
||\bar{y} - y_0 + \frac{\bar{k}}{\lambda}|| \leq \frac{1}{\lambda}.
\]
which, together with
\[ \| \bar{y} - y_0 \| > M, \]
gives a contradiction to (4.2).

If conversely (4.2) does not hold, then there exist \( \bar{y} \in S \) and \( \bar{k} \in K \) such that
\[ \| \bar{y} - y_0 \| > M \| \bar{y} - y_0 + \bar{k} \|. \]
Then there exists \( \bar{b} \in B \) such that
\[ \bar{y} - y_0 + \bar{k} = \bar{b} \| \bar{y} - y_0 + \bar{k} \| \]
and this yields
\[ s = \frac{\bar{y} - y_0}{\| \bar{y} - y_0 + \bar{k} \|} = \bar{b} - \frac{\bar{k}}{\| \bar{y} - y_0 + \bar{k} \|} \in \text{cone } (S - y_0) \cap B - K \]
and
\[ \| s \| = \frac{\| \bar{y} - y_0 \|}{\| \bar{y} - y_0 + \bar{k} \|} > M, \]
which means that (4.1) does not hold.

To conclude the proof it is enough to see that, given the point \( y_0 \in S \), there exists \( M > 0 \) such that (4.2) holds if and only if
\[ \| y + k - y_0 \| \geq L \| y - y_0 \| \]
for \( L = 1/M \) and this is equivalent to
\[ (4.4) \quad d_{-K}(y - y_0) = \inf_{k \in K} \| y + k - y_0 \| \geq L \| y - y_0 \| \quad \forall y \in S. \]

Notice at last that under both assumption that \( y_0 \in SE(S) \) or that \( \Delta_{-K} \) is nonnegative then \( d_{-K} \) coincides with \( \Delta_{-K} \), so that (4.4) proves the result.

(b) the same as in part a for \( \| y - y_0 \| \leq \eta \). \( \square \)

We recall that in Mathematical Programming a point \( x_0 \) is called a sharp minimum for the function \( f : X \to [-\infty, +\infty] \) relative to the set \( A \subseteq X \) if there exists \( \alpha > 0 \) such that it holds
\[ f(x) \geq f(x_0) + \alpha \| x - x_0 \| \]
for all \( x \in A \). This has important consequences in convergence analysis of many iterative procedures (see e.g. [25, 7] for details and references). Thus, if a point \( x_0 \) is a sharp minimum for the function \( f \), then \( f \) admits a forcing function which is linear, \( \gamma(t) = \alpha t \).

Obviously for a local sharp minimum there exists a forcing function with positive slope close to the minimum point.

On the other hand, note that the requirement that \( y_0 \) be locally superficient does not imply that \( \Delta_{-K}(\cdot - y_0) \) admits a forcing function, i.e. the largest nondecreasing function \( \gamma \) such that \( \gamma(\| y - y_0 \|) \) minorizes \( \Delta_{-K}(y - y_0) \) might be identically zero. This is made clear by the following example.

Example 4.7. Let the function \( G : \mathbb{R}^2 \to \mathbb{R} \) be defined as \( G(y_1, y_2) = y_2 + y_1 e^{y_1} \) for \( y_1 < y_2 \) and \( G(y_1, y_2) = y_1 + y_2 e^{y_2} \) for \( y_1 \geq y_2 \); take \( S = \{(y_1, y_2) : G(y_1, y_2) \leq 0\} \) and \( K = \mathbb{R}^2_+ \). The origin is the only efficient point for \( S \) and it is also locally
superefficient, but it is not strictly efficient. Indeed $S$ is asymptotically close to $-K$ as shown by the sequence $y^n = (y^n_1, y^n_2) = (-n, ne^{-n}) \in S$, with $\|y^n\| \to +\infty$.

It follows from Theorems 2.10 and 2.11 that a tightly properly efficient solution $y_0 \in S$ can be characterized in terms of the scalarized problem $(P_{y_0})$ by requiring at the same time the properties of the forcing function $\gamma$ which hold for a well-posed minimum ($\gamma$ is nondecreasing and positive outside the origin) and for a local sharp minimum ($\gamma$ has a positive slope at the origin).

**Corollary 4.8.** If the admissible point $y_0 \in S$ is tightly properly efficient, then for the problem $(P_{y_0})$ there exists a nondecreasing growth function with positive slope at the origin. If the cone $K$ has a bounded base, then the converse is true.

**5. Further Results about Strict and Proper Efficiency.** This concluding Section is devoted to a deeper analysis of the notions of (restricted) efficiency defined in Section 2. Indeed the concept of strict efficiency has only recently been introduced and can be given an equivalent description in finite dimensional spaces, which sheds new light on the geometry of the admissible region. The concept of proper efficiency has a much longer history in Vector Optimization, and the underlying idea has been ana\l\i\t\i\c\\ally described in a great number of ways. In [13] an attempt was made to classify the known definitions in three main classes, each collecting definitions which coincide in finite dimensional spaces. We will see that the notions of superefficiency, local superefficiency and tight efficiency can be seen as representative examples of the three above mentioned classes.

The reason why we restrict to finite dimensional spaces for some of the results of the present section is that we will need to assume that the ordering cone $K$ has a (weakly) compact base $\Theta$ and this is indeed true, in any Euclidean space, for any closed convex pointed cone $K$, while the same assumption proves to be very restrictive in infinite dimensional spaces (see [9]) and fails to hold for the nonnegative orthant in most common spaces.

It is immediate to verify that, if $K$ is pointed, $y_0 \in S$ is efficient exactly when $(S + K - y_0) \cap -K = \{0\}$ holds. In the case where $S$ in unbounded however the set $S + K$ needs not be closed even if both $S$ and $K$ are. Thus the condition $cl(S + K - y_0) \cap -K = \{0\}$ is a stronger requirement on $y_0$ than only efficiency. We will see that the previous condition is related to strict efficiency.

**Theorem 5.1.** If $Y$ is any normed space and $y_0 \in S$ is strictly efficient, then $cl(S + K - y_0) \cap -K = \{0\}$. If $Y$ is finite dimensional the converse is true.

**Proof.** The definition of strict efficiency can be rephrased as: for every $\varepsilon > 0$ there exists $\delta > 0$ s.t. $d_K(y - y_0) > \delta$ for every $y \in S$ with $\|y - y_0\| > \varepsilon$. Hence, if there would exist sequences $y_n \in S$, $k_n \in K$ and some $k \in K \setminus \{0\}$ such that $y_n + k_n - y_0 \to -k$, then $y_n + k + k_n - y_0 \to 0$ and hence $d_K(y_n - y_0) \to 0$; moreover $y_n - y_0$ is outside some small ball around the origin since $y_n - y_0 = k + k_n$, $k \neq 0$ and $K$ is pointed. This shows that $y_0$ is not strictly efficient.

On the other hand, if $y_0 \in S$ is not strictly efficient, then there exist $\varepsilon > 0$ and sequences $y_n \in S$ and $k_n \in K$ s.t. $\|y_n - y_0\| \geq \varepsilon$ and $\|y_n + k_n - y_0\| \to 0$. Write $k_n = \lambda_n \theta_n$ with $\lambda_n > 0$ and $\theta_n \in \Theta$ and take some $\alpha < \lambda_n$ for all $n \in \mathbb{N}$ (such a number $\alpha$ exists since $k_n \notin (\varepsilon/2)B$ and $\Theta$ is compact) to define $k'_n = (\lambda_n - \alpha)\theta_n$. Thus we obtain

$$y_n + k'_n - y_0 = y_n + k_n - y_0 - \alpha \theta_n \to -\alpha \theta \neq 0,$$

at least for some subsequence $\{\theta_{n_k}\} \subseteq \{\theta_n\}$. Hence $cl(S + K - y_0) \cap -K \setminus \{0\} \neq \emptyset$. □
Theorem 2.11. The latter is trivial since, by Theorem 5.2, the tangent cone is isotone

We come now to local superefficiency: we will show that a locally superefficient point can be described in terms of the definition of local proper efficiency given by Borwein in [5], based on the separation between the ordering cone and a local conical approximation of the feasible region. For a set $A \subseteq Y$ and $x_0 \in \text{cl} A$, we call tangent cone to $A$ at $x_0$ the set

$$ T(A, x_0) = \{ v \in X : \exists \beta_n > 0, \exists x_n \in A, x_n \to x_0 \text{ with } v = \lim_n \beta_n(x_n - x_0) \}. $$

Theorem 5.2. If $Y$ is any normed space and the point $y_0$ is locally superefficient, then it is efficient and $T(S, y_0) \cap -K = \{ 0 \}$. If $Y$ is finite dimensional the opposite relation is true.

Proof. It has been proved in [6] that if $y_0 \in SE(S)$, then $\text{cl} \text{cone} (S - y_0) \cap -K = \{ 0 \}$ (the last relation is another definition of proper efficiency, due to Borwein [5]) and hence $y_0 \in SE(S \cap (y_0 + \eta B))$ implies $\text{cl} \text{cone} [(S - y_0) \cap \eta B] \cap -K = \{ 0 \}$; the first inclusion follows from the equality $T(S, y_0) = \cap_{\eta > 0} \text{cl} \text{cone} [(S - y_0) \cap \eta B]$. To prove the converse, we should show that there exist $M > 0$ and $\eta > 0$ such that

$$ \text{cone} [(S - y_0) \cap \eta B] \cap (B - K) \subseteq MB. $$

Suppose by contradiction that there exist sequences $\alpha_n > 0$ and $s_n, y_0 \in S$ with $\|s_n - y_0\| \to 0$, $\|\alpha_n(s_n - y_0)\| \to +\infty$ and $d_K(\alpha_n(s_n - y_0)) \leq 1$. The latter implies that there exists a sequence $k_n \in K$ such that $\|\alpha_n(s_n - y_0) + k_n\| \leq 1 + 1/n$, which can be rewritten as $\alpha_n(s_n - y_0) + k_n = (1 + 1/n)b_n$, with $b_n \in B$. Moreover it holds $K = \text{cone} \Theta$, where $\Theta$ is compact, and then $k_n = \lambda_n \theta_n$ with $\lambda_n > 0$ and $\theta_n \in \Theta$. It also holds $\lambda_n \to +\infty$, because $\|k_n\| \to +\infty$ and $\Theta$ is bounded. This yields

$$ \alpha_n(s_n - y_0) = (1 + 1/n)b_n - \lambda_n \theta_n $$

and

$$ \frac{\alpha_n(s_n - y_0)}{\lambda_n} = \frac{n + 1}{n\lambda_n} b_n - \theta_n. $$

The righthand side of (5.1) converges (up to subsequences) to $-\theta \in -\Theta \subset -K$ and the lefthand side converges to an element of $T(S, y_0)$. Since $0 \notin \Theta$ we have a contradiction. \qed

Theorem 5.3. Consider the following statements:

a) The point $y_0$ is tightly properly efficient in $S$;

b) there exists an open convex set $C \subset Y$ such that $-K \setminus \{ 0 \} \subseteq C$ and $(S - y_0) \cap C = \emptyset$;

c) $T(S + K, y_0) \cap -K = \{ 0 \}$.

If $Y$ is any normed space, then it holds (a) $\Rightarrow$ (b) $\Rightarrow$ (c). If $Y$ is finite dimensional we have also that (c) $\Rightarrow$ (a) and all statements are equivalent.

Proof. To prove that (a) implies (b) it is enough to show that, since there exists an open convex set $C$ with $0 \in \partial C$, and there exists $\delta$ such that $C^c \cap (\delta B - K) \subseteq B$, then it holds $-K \setminus \{ 0 \} \subseteq C$. If indeed $k \in K \setminus \{ 0 \}$ and $-k \in C^c$, then $-\lambda k \in C^c$ for all $\lambda \geq 1$ and a contradiction arises.

For the proof that (b) implies (c) one can refer to [27]. To prove the last relation we will see that (c) implies both strict efficiency and local superefficiency, considering Theorem 2.11. The latter is trivial since, by Theorem 5.2, the tangent cone is isotone with respect to set inclusion and hence it satisfies the inclusion $T(S, y_0) \subseteq T(S + K, y_0)$. To finish suppose that $y_0 \notin \text{StE}(S)$; then there exists a sequence $y_n \in S$ such
that $y_n - y_0 \notin \epsilon B$ for some $\epsilon > 0$ and $d_-K(y_n - y_0) \to 0$, which means that there exists a sequence $k_n \in K$ with $y_n + k_n - y_0 \to 0$ and $k_n \notin (\epsilon/2)B$. Since we can always write $k_n = \lambda_n \theta_n$ with $\theta_n \in \Theta$, it follows that $\lambda_n$ does not converge to zero, i.e. there exists a subsequence (we call it again $\lambda_n$) with $\lambda_n > \beta$ for some $\beta > 0$.

Now take $\alpha_n = \|y_n + k_n - y_0\|^{1/2}$ (since $\lambda_n$ is bounded away from zero and $\alpha_n$ vanishes, it eventually holds $\alpha_n < \lambda_n$) and set $k_n' = (\lambda_n - \alpha_n)\theta_n$ to obtain $y_n + k_n' - y_0 = y_n + k_n - y_0 - \alpha_n \theta_n \to 0$ and $\alpha_n^{-1}(y_n + k_n' - y_0) \to -\theta \neq 0$. \hfill $\Box$

We observe that statement (c) in Theorem 5.3 is the definition of proper efficiency given by Borwein in [4] and that statement (b) is another definition of proper efficiency attributed to Gerstewitz in [27]. The equivalence between (b) and (c) was already proved in finite-dimensional spaces in [12]. The results proved in this section complement the ones given in [13], to which we refer for more details on proper efficiency.

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